HOMEWORK #5 SELECTED SOLUTIONS

(Section 2.3 : # 18 , Section 2.4 : #4e ,5g Section 2.5: # 3)

Section 2.3, # 18

Give an example of a nested family for each condition.

(disclaimer: There are lots of very different answers for this problem)

(a) $\bigcap_{i=1}^{\infty} A_i = [0,1]$

Let $A_i = [0, 1+\frac{1}{i}]$

(b) $\bigcap_{i=1}^{\infty} A_i = (-\infty, 1]$

Let $A_i = (-\infty, 1+\frac{1}{i}]$

(c) $\bigcap_{i=1}^{\infty} A_i = \{0, 1\}$

Let $A_i = \{0\} \cup [1, 1+\frac{1}{i})$

(d) $\bigcap_{i=1}^{\infty} A_i = \emptyset$

Let $A_i = (0, \frac{1}{i})$

(Try to check that each of my answers has the desired property.)

Section 2.4: #4e

Claim: For all $n \in \mathbb{N}$,

$$1^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2.$$

Proof: We proceed by induction. Notice taking $n=1$ gives

$$\left[\frac{n(n+1)}{2}\right]^2 = \left[\frac{1(1+1)}{2}\right]^2 = 1$$

so $1^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$ holds when $n=1$.

Now assume

$$1^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$$

for some fixed $n \in \mathbb{N}$. Then

$$\left[\frac{(n+1)((n+1)+1)}{2}\right]^2 = \left(\frac{(n+1)(n+2)}{2}\right)^2$$

$$= \left[\frac{n(n+1)}{2} + n+1\right]^2$$

$$= \left[\frac{n(n+1)}{2}\right]^2 + 2\left(\frac{n(n+1)}{2}\right)(n+1) + (n+1)^2$$

$$= \left[\frac{n(n+1)}{2}\right]^2 + n(n+1)^2 + (n+1)^2$$

$$= \left[\frac{n(n+1)}{2}\right]^2 + (n+1)^3.$$

Thus, using the induction hypothesis, we have

$$\left[\frac{(n+1)((n+1)+1)}{2}\right]^2 = \left[\frac{n(n+1)}{2}\right]^2 + (n+1)^3$$

$$= 1^3 + \cdots + n^3 + (n+1)^3$$

as desired. Thus the result holds by induction. $\blacksquare$

Section 2.4, #5q

Claim: If A has n elements, then $\mathcal{P}(A)$ has 2^n elements. $(n \in \mathbb{N})$

Proof: We proceed by induction on n.

Base Case: Let A be a set with one element, call it a. Then $A = \{a\}$ and

$$\mathcal{P}(A) = \{\emptyset, \{a\}\}$$

so $\mathcal{P}(A)$ has $2 = 2^1$ elements. Thus the claim holds for $n=1$.

Inductive step: Suppose that $n \in \mathbb{N}$ is such that for any set with n elements its power set has 2^n elements. Let A be a set with $n+1$ elements. Since $n+1>0$, A is non empty, so take $a \in A$. Let $B = A - \{a\}$, so B has n elements and thus by the inductive hypothesis $\mathcal{P}(B)$ has 2^n elements.

We claim that $\mathcal{P}(A)$ has twice as many elements as $\mathcal{P}(B)$, which will complete the proof, because then the size of $\mathcal{P}(A)$ will be

$$2(2^n) = 2^{n+1}.$$

Let $S_1 = \{C \subseteq A : a \notin C\}$, $S_2 = \{C \subseteq A : a \in C\}$.

Notice $\mathcal{P}(A) = S_1 \cup S_2$ and $S_1 \cap S_2 = \emptyset$, so to complete the proof we only have to show

S_i has the same size as $\mathcal{P}(B)$ for $i = 1, 2$.

Notice $S_1 = \mathcal{P}(B)$.

Finally, notice $C \in S_2 \iff C - \{a\} \in \mathcal{P}(B)$

so S_2 and $\mathcal{P}(B)$ have the same size.

Thus

$$\begin{aligned}
\text{size of } \mathcal{P}(A) &= (\text{size of } S_1) + (\text{size of } S_2) \\
&= (\text{size of } \mathcal{P}(B)) + (\text{size of } \mathcal{P}(B)) \\
&= 2^n + 2^n \\
&= 2^{n+1}
\end{aligned}$$

as desired. $\blacksquare$

Section 2.5, #3

Let $a_1 = 2$, $a_2 = 4$ and

$$a_{n+2} = 5a_{n+1} - 6a_n \quad \text{for } n \geq 1.$$

Then $a_n = 2^n$ $\forall n \in \mathbb{N}$.

Proof: Notice that $2^1 = 2 = a_1$ and
$2^2 = 4 = a_2$

so the claim holds for $n = 1, 2$.

Now assume $m \in \mathbb{N}$, $m > 2$ is such that $a_k = 2^k$ $\forall k \in \mathbb{N}$ with $k < m$.

We will show $a_m = 2^m$.

Notice

$$a_m = 5a_{m-1} - 6a_{m-2}$$
$$= 5(2^{m-1}) - 6(2^{m-2})$$

using the inductive hypothesis. So

$$a_m = 5 \cdot 2 \cdot (2^{m-2}) - 6(2^{m-2})$$
$$= 4 \cdot (2^{m-2}) = 2^m$$

as desired. ∎